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SELF-CONJUGATION OF SOLUTIONS VIA A SHOCK WAVE: LIMITING SHOCK

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An analytical description is given to the solution of gas-dynamic equations corresponding to two-dimensional steady gas flow involving an oblique shock. For this flow, two limiting asymptotic regimes are possible: a decelerating supersonic flow regime and a flow regime accelerating to maximum horizontal velocity. A shock solution corresponds to switching over between integral curves of the governing equation. In the case of an extremely strong shock wave, the shock becomes limiting and rotates the flow through the maximum possible angle (for an adiabatic exponent equal to three). The shock-wave structure proposed is general for a broad class of nonbarochronic, regular, partially invariant solutions of the equations of gas dynamics.

Key words: *partially invariant solution, shock wave, limiting shock.*

Introduction. Discontinuous solutions, in particular shock-wave solutions, play an important role in gas dynamics. However, the exact solutions of the equations of gas dynamics (EGD) describing shock waves are few in number [1, 2]. Most applied problems consider shock waves in homogeneous steady gas flows. Exceptions are the shock waves in self-similar one-dimensional solutions that correspond to the cylindrical and spherical symmetries of motion. Examples of gas flows with strong discontinuities belonging to the classes of planar and spatial double waves are considered in [3].

At present, the theoretical group approach is the most powerful general method for obtaining exact solutions of differential equations of various natures [4]. Its efficiency has been proved by a number of studies of mathematical models of continuum mechanics, in particular, the gas-dynamics model [5, 6]. At the same time, the search for discontinuous solutions has been insufficiently active. Investigation of this class of solutions was pioneered by Men'shchikov [7, 8], who proved the general theorems on the invariance of the characteristics and strong-discontinuity equations. From this theorems it follows that an invariant solution has an invariant continuation through an invariant characteristic or a strong-discontinuity surface. The cited papers give a number of examples.

In the present paper, we study the exact solution of the EGD that corresponds to two-dimensional steady inhomogeneous gas flows in which an oblique shock is possible. A complete analytical description is given to the solution belonging to the class of nonbarochronic, regular, partially invariant solutions of the EGD of rank one and defect one [9]. All the functions defining this solution are determined using the auxiliary function X, a peculiar solution potential. This function is a solution of the governing equation derived from the invariant Bernoulli integral. We note that this equation is not resolved for the derivative. At each point of the definition domain of this solution, its integral curves (IC) form a bundle. It was proved that two different IC corresponding to the states ahead of and behind the shock front can be chosen from this bundle so that the following conditions on the shock are satisfied: the Rankine–Hugoniot relations, the conservation of the tangential velocity component, and the Zemplén's theorem. In this case, the gas flow on both sides of the shock corresponds to a same type solution of the EGD, which can be regarded as self-conjugation of the solution through the shock wave.

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In the gas flow considered, two limiting asymptotical regimes are possible: one corresponding to a decelerating flow and the other to a flow accelerating to the maximum horizontal velocity. We also studied the geometry of the sonic line on which there is a continuous transition of the flows through the speed of sound. The flow with an oblique shock corresponds to switching over between the above flows. On the shock, the gas flow decelerates. The solution simulates inlet gas flow. In the case of an extremely strong shock wave, this shock is limiting; i.e., it rotates the gas flow through the maximum possible angle equal to $\arcsin 1/3$ (for $\gamma = 3$) [10].

The present paper reports new results, In addiiton, it has a general methodical nature. Although we study self-conjugation of a specific solution through a shock wave, the proposed method for joining different IC from a bundle is universal for a wide range of nonbarochronic, regular, partially invariant solutions of the EGD, which are described in detail in [9]. Investigation of other solutions of this class seems promising for hydrodynamic applications.

1. Initial Solution. We consider the nonbarochronic, regular, partially invariant solution of the EGD generated by the subalgebra $L_{4.38} = \{\partial_y, \partial_z, t\partial_y + \partial_v, \partial_t\}$ from the optimal system of subalgebras ΘL_{11} of Lie's algebra admitted by the EGD with an equation of state of general form [5]:

$$u = 1/\sigma', \quad v = (y + H(\xi, \eta))/\sigma, \quad w = W_0, \quad \rho = R_0\sigma'/\sigma, \quad S = S_0. \tag{1.1}$$

In Eq. (1.1), $\xi = t - \sigma(x)$, $\eta = z - tW_0$, H is an arbitrary function, and W_0, R_0, and S_0 are constants. The function $\sigma = \sigma(x)$ is determined, according to a general scheme [9], from the B-equation, i.e., the invariant Bernoulli integral

$$u^2/2 + I(\rho) = b_0, \tag{1.2}$$

where $I(\rho) = \displaystyle\int \frac{dp}{\rho}$ is the gas enthalpy and $b_0 > 0$ is a constant. Integral (1.2) is a consequence of the first momentum equation of the EGD on solution (1.1). For a polytropic gas with the equation of state $p = S_0\rho^\gamma$, relation (1.2) in terms of the function σ with allowance for formula (1.1) is written as

$$1/\sigma'^2 + \mu(\sigma'/\sigma)^{\gamma-1} = 2b_0, \tag{1.3}$$

where $\mu = 2\gamma(\gamma-1)^{-1}S_0R_0^{\gamma-1}$. Then, the Bernoulli integral (1.2) is written as

$$u^2/2 + c^2/(\gamma-1) = b_0, \tag{1.4}$$

where $c = \sqrt{\gamma p/\rho}$ is the speed of sound. From (1.4), we obtain the following estimates for u and c:

$$|u| \leqslant u_{\max} = \sqrt{2b_0}, \qquad c \leqslant c_{\max} = \sqrt{(\gamma-1)b_0}.$$

Equation (1.3) serves to determine the function $\sigma = \sigma(x)$. The solution of this equation is used to restore the remaining characteristics of the gas motion according to formulas (1.1).

Remark. In formula (1.1) for density, the constant R_0 can always be made positive (if $R_0 < 0$, the substitution $x \to -x$ is required). Then, from the nonnegativity of the density and representation (1.1) it follows that the quantities σ and σ' have the same signs. Solution (1.1) can be considered in two regions: $\sigma > 0$ and $\sigma' > 0$ and $\sigma < 0$ and $\sigma' < 0$. The first case corresponds to the gas motion in the direction of increasing x (for $u > 0$). In the second case, $u < 0$ and the gas moves from right to left. Below, we consider the first case in detail; all quantitative results are true for the second case as well.

2. Forms of the Governing Equation. Let us consider the two-dimensional steady-state solution of the EGD derived from (1.1) for $H \equiv 0$ and $W_0 = 0$ in the case of a polytropic gas:

$$u = 1/\sigma', \qquad v = y/\sigma, \qquad \rho = R_0\sigma'/\sigma, \qquad p = S_0\rho^\gamma. \tag{2.1}$$

Solution (2.1) describes steady planar flows, the gas flow is inhomogeneous, and the vortex $\omega = -y\sigma'/\sigma^2 \neq 0$. The solution belongs to the class of solutions for which the velocity field is linear in spatial variables. Such solutions have been studied previously [11].

The Lagrangian variable ψ conserved along the streamlines in the gas flow (2.1) is the velocity component v. The streamlines on the physical plane are defined by the equation

$$y = \psi\sigma(x). \tag{2.2}$$

The function $\sigma(x)$ in (2.2) is a solution of Eq. (1.3).

It is convenient to introduce a new function $X = X(x)$ using the scale transformation

$$X = (b_0/\sqrt{\mu})\sigma. \tag{2.3}$$

Then, the governing equation (1.3) becomes

$$X'^{\gamma+1} - (2b_0/\mu)X'^2 X^{\gamma-1} + (b_0^2/\mu^2)X^{\gamma-1} = 0. \tag{2.4}$$

Considering flows for which $u > 0$ and taking into account the note in Sec. 1, one can assume that $X > 0$ and $X' > 0$. In the second case for an arbitrary real adiabatic exponent $\gamma > 1$, Eq. (2.4) should be written in terms of $|X|$ and $|X'|$. For simplicity, we consider the first case in more detail. All conclusions on the nature of gas motion are true for the second case as well.

Below, we use the form of Eq. (2.4) resolved for X:

$$X^{\gamma-1} = \frac{\mu}{2b_0} \frac{X'^{\gamma+1}}{X'^2 - b_0/(2\mu)}. \tag{2.5}$$

The physical quantities (2.1) in terms of function (2.3) have the form

$$u = \frac{b_0}{\sqrt{\mu}X'}, \qquad v = \frac{b_0 y}{\sqrt{\mu}X}, \qquad \rho = R_0 \frac{X'}{X}, \qquad c^2 = \frac{\mu(\gamma-1)}{2}\Big(\frac{X'}{X}\Big)^{\gamma-1}. \tag{2.6}$$

Equation (2.4) is not resolved for the derivative. For $\gamma = 3$, for which it is a biquadratic equation for X', there is a simple formula that expresses X' in terms of X. For the other exponents γ, even integer ones, no explicit formulas are available. At the same time, for an arbitrary rational γ, the solution of Eq. (2.5) can be written in parametric form (see Sec. 5). Some properties of the solution of Eq. (2.4) can be derived by direct analysis but some facts can be proved only for $\gamma = 3$. These differences are purely technical, and, hence, insignificant. Section 3 describes the general properties of the solution of Eq. (2.4) for an arbitrary γ.

3. Properties of the Solution of the Governing Equation. We consider the governing equation in the form of (2.4) or (2.5).

Property 1. *All solutions of Eq.* (2.4), *except for* $X \equiv 0$, *are strictly monotonic functions of the variable* x.

Proof is derived from Eq. (2.4): if $X' = 0$, then $X = 0$.

Property 2. *For all solutions of Eq.* (2.4), *the following estimates are valid:*

$$|X| \geqslant X_*, \qquad |X'| \geqslant z_0. \tag{3.1}$$

Here

$$z_0 = \sqrt{\frac{b_0}{2\mu}}, \qquad z_* = \sqrt{\frac{(\gamma+1)b_0}{2(\gamma-1)\mu}}, \qquad X_* = \Big[\frac{(\gamma-1)\mu^2 z_*^{\gamma+1}}{2b_0^2}\Big]^{1/(\gamma-1)}. \tag{3.2}$$

The straight lines $X = \pm X_*$ *are not integral curves of Eq.* (2.4).

Proof. Let us introduce the auxiliary function

$$H(z) = z^{\gamma+1}/(z^2 - z_0^2). \tag{3.3}$$

Then, Eq. (2.5) takes the form

$$X^{\gamma-1} = (\mu/(2b_0))H(X'). \tag{3.4}$$

We denote $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 = \{(X', X)\colon\ X' \geqslant z_0,\ X > X_*\}$ and the region $\Omega_2 = \{(X', X)\colon X' \leqslant -z_0,\ X < -X_*\}$ is located in the third quarter. Function (3.3) in Ω_1 for $z > z_0$ has a local minimum at $z = z_*$ ($z_* > z_0$ by virtue of $\gamma > 1$). Then, X_* is determined from (3.4):

$$X_* = [(\mu/(2b_0))H(z_*)]^{1/(\gamma-1)}.$$

We have

$$H(z) \underset{z \to z_0 + 0}{\longrightarrow} +\infty, \qquad H(z) \underset{z \to +\infty}{\longrightarrow} +\infty. \tag{3.5}$$

Estimates (3.1) are proved. Figure 1 shows the characteristic curve (3.4) with allowance for (3.5) in the region Ω_1. We denote this curve by Γ_0, the minimum point by A, and the branches on the left and right of this point by Γ_{01} and Γ_{02}, respectively. We note that the motion along the curve Γ_0 from the point A both to the left along the branch Γ_{01} and to the right along the branch Γ_{02} corresponds to an increase in X and, according to Property 1, to an increase in the x coordinate.

Property 3. *Through each point on the plane* $\mathbb{R}^2(x, X)$ *there pass not more than two integral curves of Eq.* (2.5) *for which* $XX' > 0$.

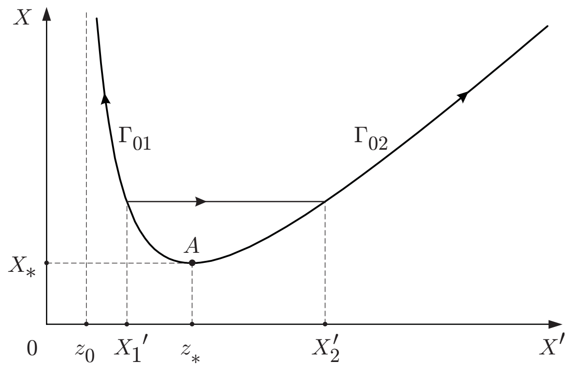

Fig. 1

Proof. According to Property 2, for any X ($|X| > X_*$) in the region Ω, Eq. (2.5) has two solutions: X_1' and X_2' (Fig. 1); i.e., the function $H(z)$ (3.3) is double valued for $|z| > z_0$. By virtue of the monotonic dependence of the function X on x (Property 1), this is also valid with the replacement of X by x. Hence, for x such that $|X(x)| > X_*$, each point has a bundle of two integral curves of Eq. (2.5) satisfying the condition $X' > 0$ (this condition distinguishes the region Ω). These curves differ in the values of X_i' ($i = 1, 2$), i.e., in slopes. Thus, all IC on the plane $\mathbb{R}^2(x, X)$ lie above the straight line $X = X_*$ and below $X = -X_*$, and these boundaries are not IC.

Property 4. *There exist two asymptotic regimes that correspond to continuous gas motions as* $x \to +\infty$:

1) *A gas flow decelerating along the* Ox *axis, in which maximum density and maximum speed of sound* $c_{\max}$ *are attained*:

$$\rho \to R_0\alpha_0, \qquad c \to c_{\max}, \qquad u \to 0 \qquad [\alpha_0 = (2b_0/\mu)^{1/(\gamma-1)}]; \tag{3.6}$$

2) *A gas flow accelerating along the* Ox *axis, which rarefies with increasing* x:

$$\rho \to 0, \qquad c^2 \to 0, \qquad u \to u_{\max}. \tag{3.7}$$

Proof. According to Eq. (2.5), we have

$$\Big(\frac{X'}{X}\Big)^{\gamma+1} = \frac{2b_0}{\mu}\Big(\frac{X'}{X}\Big)^2 - \frac{b_0^2}{\mu^2}\,\frac{1}{X^2}. \tag{3.8}$$

As $x \to +\infty$, we have $X \to +\infty$ (Property 1); then, from Eq. (3.8) it follows that

$$\alpha^{\gamma+1} = (2b_0/\mu)\alpha^2, \tag{3.9}$$

where $\alpha = \lim\limits_{x\to\infty}(X'/X)$. By virtue of representation (2.6) and Bernoulli integral (1.4), the quantity α is limited and is a real number. Equation (3.9) has two solutions: $\alpha = \alpha_0$ and $\alpha = 0$.

1. Let $\alpha = \alpha_0$. Then, from formulas (2.6) and (1.4), we obtain the asymptotic relation (3.6). As $x \to \infty$, the slope of the integral curve increases unboundedly with increasing X, and

$$X'/X \to (2b_0/\mu)^{1/(\gamma-1)}, \qquad x \to +\infty. \tag{3.10}$$

This flow (supersonic decelerating gas flow in which the density and the speed of sound reach maximum values) corresponds to the branch Γ_{01} of the curve Γ_0 in Fig. 1.

2. Let $\alpha = 0$. According to (2.6) and (1.4), the limiting values of the density and the speed of sound are zero; i.e., we have regime (3.7). The gas accelerates in the Ox direction with simultaneous rarefaction; the flow is subsonic. The slope of the integral curve corresponding to this regime tends to the constant value:

$$X' \to (\sqrt{\mu}/b_0)u_{\max} = \sqrt{2\mu b_0}, \qquad x \to +\infty. \tag{3.11}$$

The IC has an asymptote, which is a linear function. The branch Γ_{02} in Fig. 1 corresponds to this IC.

Thus, the gas flow starting at the point A corresponds to the different branches (Γ_{01} and Γ_{02}) of curve (3.4), depending on the conditions at infinity (3.6) or (3.7). Since the governing equation (2.4) is not resolved for the derivative, for unique resolution of the problem with the initial data, one need to specify not only the function value but also one of the possible values of the derivative at a specified point. These are conditions (3.10) or (3.11), which correspond to continuous gas motions without strong or weak discontinuities. There is an analogy with a gas source with two regimes.

Property 5. *The integral curve $X = X(x)$ of Eq.* (2.4) *corresponding to the branch Γ_{01} is convex downward and that corresponding to the branch Γ_{02} is convex upward.*

Proof. Differentiation of relation (2.5) for $(X, X') \in \Omega_1$ yields

$$(2b_0/\mu)X^{\gamma-2}(X')^{1-\gamma}(X'^2 - z_0^2)^2 = (X'^2 - z_*^2)X''. \tag{3.12}$$

The left side of (3.12) is nonnegative in Ω_1; hence, we have $X'' < 0$ for $X' < z_*$ and $X'' > 0$ for $X' > z_*$. [Here, we consider the convexity of the curves on the plane $\mathbb{R}^2(x, X)$.]

Property 6. *The straight lines $X = \pm X_*$ are the limiting trajectories of motion* (2.6): *on these trajectories, the gas-flow acceleration in the Ox direction becomes unbounded; the solution does not extend beyond these trajectories.*

Proof. We consider representation (3.12) for X''. As $X \to X_* + 0$, we have $X' \to z_*$. On the left side of Eq. (3.12), we obtain a finite quantity, and on the right side, we have the product of a small quantity into X'', and, hence, $X'' \to \infty$ as $X \to X_* + 0$. These straight lines, i.e., the point A in Fig. 1, correspond to an injection of inhomogeneous flow in a certain cross section $x = x_1$. The flow parameters in this cross section are determined by the coordinates of the point A (see Sec. 13).

4. Bernoulli Integral and Streamlines. The equation of streamlines (2.2) in terms of the function X (2.3) is written as

$$y = \psi\sqrt{\mu}X(x)/b_0. \tag{4.1}$$

The variable ψ in Eq. (4.1) "numbers" the streamlines, thus determining the point on the plane $\mathbb{R}^2(x, y)$ through which these streamlines pass. In (4.1), X is a solution of Eq. (2.4).

Besides the invariant Bernoulli integral (1.4), there is a general integral for this flow:

$$(u^2 + v^2)/2 + c^2/(\gamma - 1) = b_0 + b_0^2 y^2/(2\mu X^2). \tag{4.2}$$

Formulas (2.6) and (4.2) define the isentropic form of this solution. Using Munk–Prim's transformation [12], one obtains the isoenergetic but not the isentropic form, in which the right side of the Bernoulli integral (3.2) is constant but the entropy in the equation of state is a function of the variable ψ. This reduction is performed by dividing relation (4.2) by its right side, which depends on the Lagrangian coordinate ψ alone. The formulas

$$U = u/B(\psi), \qquad V = v/B(\psi), \qquad R = B^2(\psi)\rho, \qquad P = p, \tag{4.3}$$

where $B^2(\psi) = b_0 + (b_0^2/(2\mu))\psi^2$, specify the above transition. The trajectory of the gas motion described by Eq. (4.3) coincides with the trajectory of the initial motion (2.1) because some streamlines of these two motions coincide. The isoenergetic form is sometimes preferred in solving various hydrodynamic problems [13]. For our purposes, the isentropic form of solution (2.1) [or (2.5)] is more convenient. However, this solution is not isentropic. After transformation (4.3), the equation of state $p = S_0\rho^\gamma$ becomes the equation

$$P = S_0 B^{-2\gamma}(\psi)R^\gamma,$$

where the function B is defined above and specifies entropy. Among recent studies in which Munk–Prim's transformation is used to solve specific gas-dynamic problems, we note work of Guvernyuk [14].

5. Parametric Representation of the Solution. The solution of Eq. (2.5) is representable in parametric form for an arbitrary rational adiabatic exponent $\gamma = k/r$, where k and r are natural numbers ($k > r$). We set $q = \sqrt{2\mu/b_0}X'$; then,

$$X = X_0 q^{1+2æ}/(q^2 - 1)^{æ}, \tag{5.1}$$

where $æ = 1/(\gamma - 1)$ and $X_0 = 2^{-(2æ+1)/2}(b_0/\mu)^{1-2æ}$. The dependence $x = x(q)$ is found by integrating the relation $dx = \sqrt{2\mu/b_0}\, q^{-1}X_q^{-1}\, dq$. Using Chebyshev's theorem on integration of a differential binomial [15], we obtain

$$x = X_0\sqrt{\frac{\mu}{2b_0}}\Big[r\int\frac{dt}{t^{k+1}(1 - t^r)} - \frac{2k + r}{k}\,\frac{1}{t^k}\Big] + x_0, \tag{5.2}$$

where $t = [(q^2 - 1)/q^2]^{1/r}$ and x_0 is the integration constant. For specific values of k and r, the integral (5.2) is expressed as elementary functions. As an example, we give formula (5.2) for $\gamma = 3$:

$$x = \sqrt{\mu/(2b_0)}\,[\ln|q + \sqrt{q^2 - 1}| + |q|/\sqrt{q^2 - 1}\,] + x_0. \tag{5.3}$$

The dependence of thermodynamic quantities on x is determined from Eq. (2.6) and (5.1).

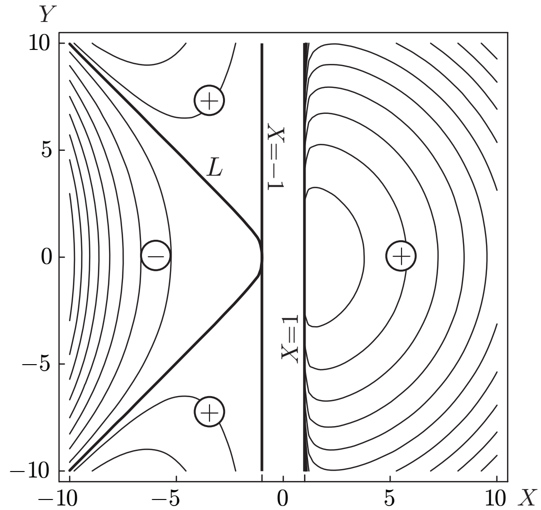

Fig. 2

6. Sonic Line ($\gamma = 3$). We determine the type of the sonic line in (2.6) for $\gamma = 3$. Substituting representation (2.6) into the equation and using (2.4), we obtain that on the plane $\mathbb{R}^2(X, Y)$ [$Y = \sqrt{b_0/(2\mu)}\, y$ and $X = X(x)$ is a solution of Eq. (2.5)], the supersonic flow region is defined by the inequality

$$X[(X + \varepsilon\sqrt{X^2 - 1})(Y^2 - X^2) + X] > 0, \qquad \varepsilon = \pm 1. \tag{6.1}$$

In this case, $|X| > 1$. We note that inequality (6.1) is preserved as $X \to -X$ and $\varepsilon \to -\varepsilon$. Hence, it suffices to solve this inequality for at least one of the values of ε, for example, for $\varepsilon = -1$.

It turns out that for $\varepsilon = -1$, the flow is supersonic over the entire half plane $X > 1$ and in the infinite "sectors" bounded by the straight line $X = -1$ and the branches of the curve L: $Y^4 = X^4 - X^2$. Its asymptotes are the straight lines $Y = \pm X$. In Fig. 2, the supersonic flow regions are denoted by the plus sign and the subsonic flow regions, by the minus sign.

The streamlines (4.1) on the plane $\mathbb{R}^2(X, Y)$ correspond to the straight lines $Y = kX$. During the gas motion in the region $X < -1$, there may be a change of the flow type: the supersonic flow continuously transforms into a subsonic flow. To depict the sonic line on the physical plane $\mathbb{R}^2(x, y)$, it is necessary to specify the dependence of the solution of Eq. (2.4) on the variable x [$X = X(x)$].

7. Flows with a Steady Shock Wave. Let p_i, ρ_i, and c_i be the pressure, density, and speed of sound, and let u_i be the gas velocity components normal to the front ahead of the wave front ($i = 1$) and behind the wave front ($i = 2$). In this case, the Rankine–Hugoniot conditions [16] on the shock holds:

$$\rho_1 u_1 = \rho_2 u_2; \tag{7.1}$$

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2; \tag{7.2}$$

$$\frac{\gamma}{\gamma - 1}\,\frac{p_1}{\rho_1} + \frac{1}{2}\,u_1^2 = \frac{\gamma}{\gamma - 1}\,\frac{p_2}{\rho_2} + \frac{1}{2}\,u_2^2. \tag{7.3}$$

In transition through the shock, the velocity components tangential to the wave front are conserved:

$$\boldsymbol{u}_{1\tau} = \boldsymbol{u}_{2\tau}. \tag{7.4}$$

The Zemplén theorem holds: the absolute value of the velocity component normal to the front u_i is larger than the speed of sound c_i ahead of the shock front and is smaller than that behind the shock front:

$$u_1^2 > c_1^2, \qquad u_2^2 < c_2^2. \tag{7.5}$$

This theorem is equivalent to the statement that the enthalpy increases in the shock transition.

The Bernoulli integral preserves its value during the shock transition.

We write conditions (7.1)–(7.5) for solution (2.6). Let us show that the different integral curves corresponding to the solution of Eq. (2.4) and passing through a specified point correspond to the states ahead of (state 1) and

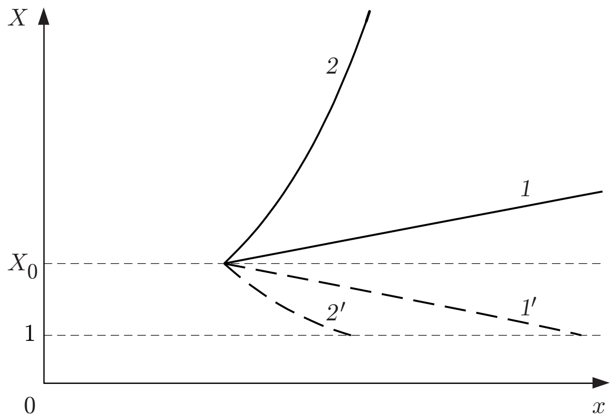

Fig. 3

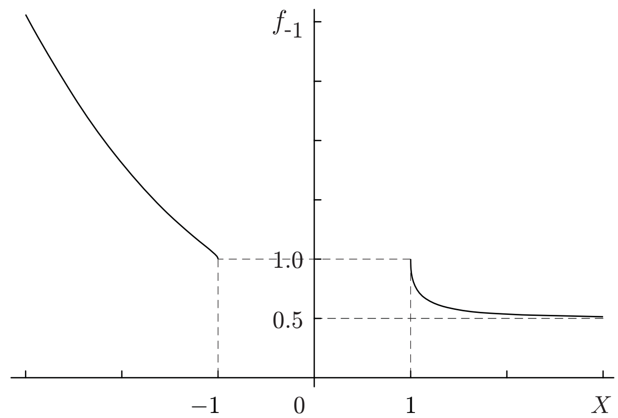

Fig. 4

behind (state 2) the shock-wave front defined by the equation $x = x_0$. Since all physical quantities are expressed in terms of the function X and its derivative X' by formulas (2.6), the shock relations (7.1)–(7.4) can be satisfied by an appropriate choice of various IC from the bundle. At the wave front, these relations link constant quantities by virtue of the dependence $X = X(x)$.

The Zemplén theorem [inequality (7.5)], defining which IC correspond to the gas state ahead of and behind the shock, has been proved for an arbitrary value of γ. The remaining relations (7.1)–(7.4) are analyzed for $\gamma = 3$.

8. Zemplén's Theorem. The gas states ahead of and behind the shock front $x = x_0$ are accounted for by the solutions X_1 and X_2 of the governing equations and the derivatives X_1' and X_2' of these solutions.

Lemma 1. *The branch Γ_{01} corresponds to the gas state ahead of the shock-wave front, and the branch Γ_{02} to that behind the shock-wave front.*

Proof. We write the first condition of (7.5) in terms of solution (2.6):

$$\frac{u^2}{c^2} = \frac{2b_0^2}{\mu^2(\gamma-1)}\,\frac{X^{\gamma-1}}{X'^{\gamma+1}} > 1. \tag{8.1}$$

Substitution of the representation for $X^{\gamma-1}/X'^{\gamma+1}$ from Eq. (2.5) into (8.1) yields

$$X'^2 < \frac{b_0}{\mu}\,\frac{\gamma+1}{2(\gamma-1)} = z_*^2. \tag{8.2}$$

Proof of Lemma 1 follows from inequality (8.2) and Properties 2 and 4. Selection is performed by the slope of the integral curves: for $X > 0$ and $X' > 0$, the state ahead of the front is represented by a curve with the smaller limited slope, and that behind the front is represented by an integral curve with the larger slope [see inequalities (3.10) and (3.11)]. We note that the state ahead of the front, i.e., the branch Γ_{01}, corresponds to an apriori supersonic flow, and the postshock flow can be either subsonic or supersonic [see Eq. (6.1)].

9. Properties of the Solution ($\boldsymbol{\gamma = 3}$). For $\gamma = 3$, it is possible to refine some properties of the solution of Eq. (2.4). Solving it for X', we obtain

$$X'^2 = (b_0/\mu)(X^2 + \varepsilon X\sqrt{X^2-1}\,), \qquad \varepsilon = \pm 1. \tag{9.1}$$

Since Eq. (9.1) is invariant under the replacements $X \to -X$ and $\varepsilon \to -\varepsilon$ (see section **6**), it suffices to study the behavior of the solutions for $\varepsilon = -1$ and $X > 0$.

1. All integral curves of Eq. (9.1), except for the straight line $X = 0$, which are monotonic curves, are located on the plane $\mathbb{R}^2(x, X)$ in the region

$$D = \{(x, X)\colon |X| > 1\}. \tag{9.2}$$

The boundaries of the region D — the straight lines $X = \pm 1$ — are the boundary lines of this flow; on these lines, X'' becomes infinite.

2. A bundle of two pairs of integral curves symmetric about the straight line $X = X_0 > 1$ (Fig. 3) passes through each point of the region D. In the solution of Eq. (9.1) for X', the plus sign corresponds to the pair of curves above this straight line. These curves account for monotonically increasing solutions, and, hence, for these

curves, $X' > 0$. Curve 2 of this pair corresponds to $\varepsilon = 1$ in formula (9.1) and curve 1 to $\varepsilon = -1$. Below the straight line $X = X_0$ there are curves $1'$ and $2'$ obtained by reflection of curves 1 and 2 in this straight line. Only the upper pair of curves satisfies the condition $XX' > 0$. Curve 1 corresponds to the branch Γ_{01}, and curve 2 to the branch Γ_{02} (see Fig. 1). Each point on the boundary straight line $X = 1$ (point A in Fig. 1) corresponds to two integral curves on which $X' = \varepsilon\sqrt{b_0/\mu}$. One of them (for $\varepsilon = 1$) goes upward from a point on the boundary, and the other (for $\varepsilon = -1$) enters this point, i.e., decreases monotonically along the variable x.

A similar picture takes place for $X < 0$ and $X' < 0$. The integral curves of Eq. (9.1) are shown in Fig. 3. The different nature of convexity of curves 1 and 2 is proved in Property 5.

3. Condition (8.2) of the Zemplén theorem, defining the gas state ahead of the front, reduces to the following inequality through the use of (9.1):

$$f_\varepsilon(X) \equiv X^2 + \varepsilon X\sqrt{X^2 - 1} < 1, \qquad \varepsilon = \pm 1. \tag{9.3}$$

Let $\varepsilon = -1$. Then Eq. (9.3) holds for $X > 1$ and does not hold for $X < -1$.

Let $\varepsilon = 1$. Then Eq. (9.3) holds for $X < -1$ and does not hold for $X > 1$.

We have

$$\lim_{X\to+\infty} f_1(X) = +\infty, \qquad \lim_{X\to-\infty} f_{-1}(X) = +\infty,$$

$$\lim_{X\to-\infty} f_1(X) = 1/2, \qquad \lim_{X\to+\infty} f_{-1}(X) = 1/2.$$

In addition,

$$f'_\varepsilon(X) = \varepsilon(X^2 - 1)^{-1/2}(X + \varepsilon\sqrt{X^2 - 1}\,)^2.$$

The function $f_\varepsilon(X)$ for $\varepsilon = -1$ is plotted in Fig. 4.

10. Shock Relations ($\gamma = 3$). Let a pair (X_1, X_1') correspond to the gas state ahead of the shock $x = x_0$ and (X_2, X_2') to the gas state behind the shock. In this case, X_i are solutions of Eq. (9.1) with $\varepsilon_1 = -1$ ($i = 1$) and $\varepsilon_2 = 1$ ($i = 2$). According to Eq. (2.6), the physical quantities also differ in values of the constants b_{0i}, R_{0i}, and S_{0i} ($i = 1, 2$). We write condition (7.4), substituting the value of v_i from (2.6):

$$(b_0 y/(\sqrt{\mu}X))_1 = (b_0 y/(\sqrt{\mu}X))_2. \tag{10.1}$$

Comparing Eq. (10.1) and the condition of conservation of the right side of the Bernoulli integral (4.2) on the shock, we find that the constant b_0 is conserved in transition through the shock: $b_{01} = b_{02} = b_0$.

Next, comparison of Eq. (10.1) and the streamline equation (4.1) shows that Eq. (10.1) implies conservation of the quantity ψ (streamline"number") in shock transition. The streamline undergoes bending at the shock, and a gas particle that crosses this streamline changes velocity. This is possible by virtue of the properties of Eq. (9.1). At each point of the region D (9.2), we have a bundle of two IC, which correspond to different streamlines. At the shock, there is switching from one IC into the other, and the gas particle continuously passes from curve 1 to curve 2 (see Lemma 1), changing velocity. Cancelling the factors with equal values on both sides of the shock on the left and right sides of (10.1), we obtain

$$\sqrt{\mu_1}X_1 = \sqrt{\mu_2}X_2. \tag{10.2}$$

Equations (2.6) for the states on both sides from the shock are written as

$$\begin{gathered} u_i = b_0/(\sqrt{\mu_i}X_i'), \qquad \rho = R_{0i}X_i'/X_i, \qquad p_i = S_{0i}R_{0i}^3(X_i'/X_i)^3, \\ c_i^2 = \mu_i(X_i'/X_i)^2, \qquad \mu_i = 3S_{0i}R_{0i}^2, \qquad i = 1, 2. \end{gathered} \tag{10.3}$$

Below, we consider the region $\Sigma_1 = \{(X_1, X_1'; X_2, X_2')\colon\ X_1 \geqslant 1,\ X_1' > 0,\ X_2 \geqslant 1, \text{ and } X_2' > 0\}$. The equation of conservation of mass at the shock (7.1) is written using (10.3):

$$\sqrt{S_{01}}X_1 = \sqrt{S_{02}}X_2. \tag{10.4}$$

Substituting expressions for μ_i from (10.3) into (10.2), we arrive at Eq. (10.4) provided that

$$R_{01} = R_{02} = R_0. \tag{10.5}$$

One of the reduced conditions at the shock is condition (10.4). The condition of energy conservation at the shock Eq. (7.3) coincides with the invariant Bernoulli integral (1.4), and, hence, with Eq. (2.4). It does not impose additional restrictions (conservation of the constant b_0 follows from the general Bernoulli integral). Thus, the governing equation (2.4) [or Eq. (9.1) for $\gamma = 3$] is condition (7.3) at the shock.

Reducing the condition of conservation of momentum at the shock (7.2) to canonical form is more intricate compared to the reduction of relations (7.1) and (7.3). The expression $p + \rho u^2$ is transformed by substituting representation (2.6) into it, replacing the derivatives from (10.1), and reducing it using (10.4) and (10.5). The expression for the shock ($[f] = f_2 - f_1$) is transformed as follows:

$$[p + \rho u^2] = \Big[\frac{S_0 R_0^3}{XX'}\Big(\frac{X'^4}{X^2} + \frac{b_0^2}{\mu S_0 R_0^2}\Big)\Big] = \Big[\frac{S_0^{3/2}}{(\sqrt{S_0}X)X'}\Big(X'^2 + \frac{2b_0}{\mu}\Big)\Big] = \Big[\frac{S_0^{3/2}}{\mu^2 X'}(X^2 + \varepsilon X\sqrt{X^2-1} + 1)\Big]. \tag{10.6}$$

The following notation is introduced:

$$A_i = X_i^2 + \varepsilon_i X_i\sqrt{X_i^2 - 1}, \qquad i = 1, 2. \tag{10.7}$$

According to Zemplén's theorem (see Lemma 1), $\varepsilon_1 = -1$ and $\varepsilon_2 = 1$. We write the shock condition (10.6) in terms of (10.7):

$$(A_1 + 1)/(\sqrt{S_{01}}X_1') = (A_2 + 1)/(\sqrt{S_{02}}X_2'). \tag{10.8}$$

Squaring equality (10.8) and substituting the values of $X_i'^2$ from (9.1), we obtain the equality

$$A_2(A_1 + 1)^2 = A_1(A_2 + 1)^2,$$

from which it follows that

$$(A_1 - A_2)(A_1A_2 - 1) = 0. \tag{10.9}$$

The condition $A_1 = A_2$ yields the equality $X_1^2 = X_2^2$; then, from (10.4), we have $S_{01} = S_{02}$, which is in conflict with the condition of increased entropy in transition through the shock. Hence, Eq. (10.9) is satisfied only if $A_1A_2 = 1$. This relation in expanded form is written as

$$\Big(X_1^2 - X_1\sqrt{X_1^2 - 1}\Big)\Big(X_2^2 + X_2\sqrt{X_2^2 - 1}\Big) = 1 \tag{10.10}$$

is an analogue of the Prandtl equation at the shock [10].

We summarize the results as the following statements.

Theorem 1. *Conditions* (7.1)–(7.3) *at the shock for solution* (10.3) and (9.1) *are equivalent to the finite relations* (10.4) *and* (10.10)*, which link the values of the solutions* X_1 *and* X_2 *of the differential equation* (9.1) *for* $\varepsilon_1 = -1$ *and* $\varepsilon_2 = 1$ *at the shock front* $x = x_0$.

Relation (10.4) defines a one-parameter family of straight lines with an angular coefficient $k = \sqrt{S_{01}/S_{02}}$, $0 < k < 1$ on the plane of state $\mathbb{R}^2(X_1, X_2)$. Equation (10.10) does not include parameters and defines a certain curve on this plane. Each straight line (10.4) corresponds to a class of shock transitions [self-conjugation of solutions of type (10.3)] with a fixed ratio $S_{01}/S_{02} = k^2$. The points of intersection of this line with curve (10.10) specify the pairs of states (X_1, X_2) that can be conjugated through the shock wave. Thus, curve (10.10) is a shock adiabat, characterizing possible shock transitions for a given solution (10.3).

11. Analysis of the Shock Adiabat (10.10). We note that Eq. (10.10) is invariant under replacement T: $X_1 \to -X_2$ and $X_2 \to -X_1$ (for $T^2 = 1$) (involution). Therefore, this curve can be considered for $X_1 \geqslant 1$ and $X_2 \geqslant 1$. We introduce the parametrization

$$X_1 = 1/\sin\varphi, \qquad X_2 = 1/\sin\psi,$$

where $\varphi \in (0, \pi/2]$ and $\psi \in (0, \pi/2]$. Then, Eq. (10.10) is brought to the form

$$\cos(\varphi/2)\sin(\psi/2) = 1/2. \tag{11.1}$$

The functions on the left side of relation (11.1) are monotonic on the indicated intervals. We have $\psi = \pi/2$ for $\varphi = \pi/2$, and $\psi \to \pi/3 + 0$ as $\varphi \to +0$. Hence, $\psi \in (\pi/3, \pi/2]$. Equation (11.1) is uniquely solvable in the form

$$\varphi = 2\arccos(1/(2\sin(\psi/2))). \tag{11.2}$$

Relation (11.2) defines φ as a single-valued function in the specified interval. An important feature of this curve is its asymptotic behavior. If $X_1 \to +\infty$ ($\varphi \to +0$), then

$$X_2 \to X_{2l} = 2/\sqrt{3} \approx 1.1574 \qquad (\psi \to \pi/3 + 0). \tag{11.3}$$

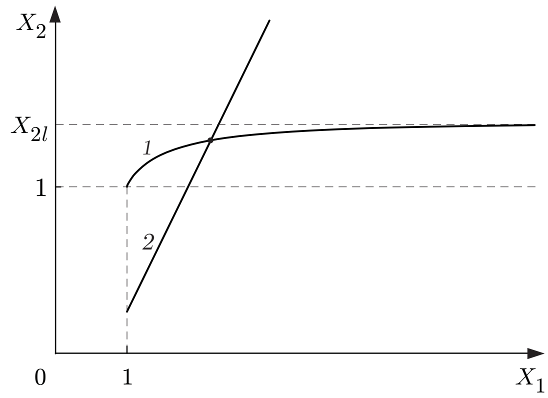

Fig. 5

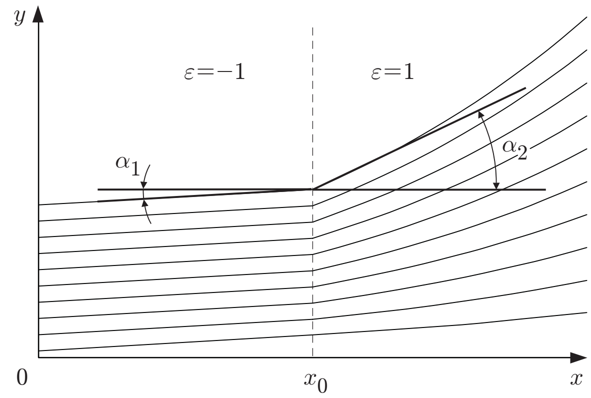

Fig. 6

Curve (10.10) is shown in Fig. 5 (curve 1). Straight line (10.4) (curve 2 in Fig. 5) has a single point of intersection with curve (10.10) for $S_{01}/S_{02} \in (0, 1)$. This follows from the fact that curve (10.10) passes through the point $X_1 = X_2 = 1$, increases strictly monotonically, and $X_2 \leqslant X_{2l}$ on this curve. We substitute $X_2 = kX_1$ into Eq. (10.10) and set $X_1 = 1/\sin\varphi$ and $\varphi \in (0, \pi/2]$. Then, Eq. (10.10) becomes

$$P(z) \equiv z^4 - z^3 + (k^2/4)z - k^2/16 = 0,$$

where $z = \cos^2(\varphi/2)$ and $z \in [1/2, 1]$. Because $P(1/2) < 0$ and $P(1) > 0$, it follows that $P(z)$ has a real root on this interval. Due to the monotonicity and boundedness of curve (10.10), this root is unique.

Thus, any state ahead of the shock (X_1, S_{01}) corresponds to a pair (X_2, S_{02}), where X_2 is calculated from Eq. (10.10), and S_{02} is calculated according to Eq. (10.4).

12. Third Limiting Flow Regime. According to Eq. (9.1), the limiting state $X_2 = X_{2l}$ (11.3) corresponds to $X'_{2l} = \sqrt{2b_0/\mu_2}$. The physical parameters of this flow are calculated by formulas (10.3):

$$u_{2l} = \varepsilon\sqrt{b_0/2}, \qquad c_l = \sqrt{3b_0/2}, \qquad \varepsilon = \pm 1. \tag{12.1}$$

The velocity component v_l is calculated from formulas (2.6). The regime (12.1) is asymptotic as $X_1 \to +\infty$ and $X_2 \to X_{2l}$. Then, it follows from Eq. (10.4) that $S_{02}/S_{01} \to +\infty$; i.e., the shock wave is strong.

We note one more property of the limiting flow (12.1).

Theorem 2. *In the case of a strong shock wave, shock transition on solution* (10.3) *causes an asymptotically oblique shock involving flow rotation through the maximum possible angle.*

Proof. The streamlines on the physical plane are defined by Eq. (4.1). We denote the angles formed by the tangents to the streamlines ahead of and behind the shock by α_1 and α_2, and the flow rotation angle by $\Delta = \alpha_2 - \alpha_1$ (Fig. 6). Then,

$$\tan\,\alpha_i = (\sqrt{\mu_i}\psi/b_0)X'_i(x), \qquad i = 1, 2. \tag{12.2}$$

Substituting X'_i from (9.1) into (12.2), we have

$$\tan\,\Delta = \frac{\psi}{\sqrt{b_0}}\,\frac{\sqrt{A_2} - \sqrt{A_1}}{1 + (\psi^2/b_0)\sqrt{A_1 A_2}}, \tag{12.3}$$

where A_i are defined by formulas (10.7). Condition (10.10) at the shock has the form $A_1 A_2 = 1$, where $A_1 \geqslant 0$ and $A_2 \geqslant 0$. By virtue of this condition, expression (12.3) becomes

$$\tan\,\Delta = K_0 F(X_2), \tag{12.4}$$

where the constant is

$$K_0 = 1/(\sqrt{b_0}/\psi + \psi/\sqrt{b_0}) \leqslant 1/2, \tag{12.5}$$

and the function $F(X_2) = \sqrt{A_2} - (1/\sqrt{A_2})$ increases strictly monotonically for $X_2 > 1$. Indeed,

$$F'(X_2) = (1 + 1/A_2^2)\Big(X_2 + \sqrt{X_2^2 - 1}\Big)^2 \Big/ \sqrt{X_2^2 - 1} > 0.$$

Hence, the function $F(X_2)$ reaches a maximum value for the maximum possible value of the argument, i.e., as $X_2 \to X_{2l}$:

$$F(X_2) \leqslant F(X_l) = 1/\sqrt{2}. \tag{12.6}$$

Taking into account Eq. (12.5) and (12.6), we obtain the following estimate for Eq. (12.4):

$$\tan\Delta \leqslant 1/(2\sqrt{2}\,). \tag{12.7}$$

It follows from Eq. (12.7) that $\sin\Delta = 1/3$. This coincides with the value of $1/\gamma$, characterizing the limiting angle of flow rotation in an oblique shock [10]. Flow (12.1) occurs behind the shock.

We also have

$$\lim_{X_1\to+\infty} A_1 = 1/2, \qquad \lim_{X_2\to X_{2l}} A_2 = 2, \tag{12.8}$$

where A_i are the monotonic functions of the arguments X_i (see Fig. 4). Equations (12.8) and (9.1) yield the estimates

$$X_1' \leqslant æ_0/(\sqrt{2}S_{01}), \qquad X_2' \leqslant \sqrt{2}æ_0/S_{02}, \qquad æ_0 = \sqrt{b_0/(3R_0^2)}.$$

13. Physical Flow Pattern ($\gamma = 3$). For the flow (2.6), the straight lines $X = \pm 1$ are the limiting lines (see Property 6), and the solution does not extend beyond them. These straight lines can be regarded as the lines on which the inlet gas flow with the following parameters is specified:

$$u = \varepsilon_1\sqrt{b_0}, \qquad c^2 = b_0, \qquad \varepsilon_1 = \pm 1; \tag{13.1}$$

the velocity component v is defined by Eq. (2.6). The flow velocity profile is linear in the vertical coordinate y, and the flow is supersonic everywhere except for the Ox axis, on which $|\boldsymbol{u}| = c$ (sonic line). On the plane $\mathbb{R}^2(X', X)$, the place of flow inlet is denoted by the point A, which corresponds to the state (13.1) of the curve Γ_0 (see Fig. 1). This point separates the branches Γ_{01} and Γ_{02}. Depending on the conditions at infinity (3.6) or (3.7), there may be two types of gas flow: a supersonic gas flow, corresponding to the branch Γ_{01} and the integral curve 1 in Fig. 3, and a subsonic flow as $x \to +\infty$, corresponding to the branch Γ_{02} and integral curve 2. According to the results of Sec. 6, a continuous transition through the speed of sound is possible in this flow.

A gas flow with a shock wave — an oblique shock decelerating the flow — is also possible. This flow also starts from the state (13.1) corresponding to the point A. The supersonic flow region before the shock corresponds to the branch Γ_{01} and curve 1, after which there is a jumpwise transition to the branch Γ_{02} and curve 2. The shock is shown by a horizontal arrow in Fig. 1. Figure 6 shows the flow pattern on the physical plane [see relations (5.1) and (5.3)]. This solution can be treated as a gas flow decelerated by an oblique shock in the air inlet.

The position of the front (value of x_0) is determined from the data ahead of the front, i.e., from the value of X_1. By virtue of the monotonic dependence $X = X(x)$, this correspondence is unique on the integral curves 1 and 2 in Fig. 3. The position of the front x_0 is first calculated from a specified value of X_1, and X_2 is then found from the shock adiabat (10.10). After this, the postshock entropy S_{02} is obtained from relation (10.4) with a specified entropy value ahead of the shock S_{01}.

Any streamline can be treated as a rigid wall. With this approach, the solution is flow in a channel with curvilinear walls, whose curvature changes jumpwise with passage of a shock front.

"Inverse" treatment of the result obtained is also possible. A jumpwise change of the curvature of a curvilinear channel results in the formation of a shock wave in the channel.

14. Discussion of Results. The shock-transition structure considered above can be applied to general solutions of the form of (1.1), where $H \not\equiv 0$. This is possible because the arguments H are Lagrangian coordinates and are continuous during transition through a shock $x = x_0$. The addition of H changes the velocity component tangential to the front, but its value does not changes during shock transition. In this case, the solution is unsteady and the wave front $x = x_0$ is stationary.

Of interest is the problem of constructing a shock for the maximally general equation of gas state. In this case, in the governing equation (1.2), the nonlinear dependence on the derivative is defined by the form of the equation of state, i.e., by the function $I = I(\rho)$. A bundle can, in principle, contain an arbitrary number of IC. The question arises of whether a sequence (cascade) of shocks is realizable.

Extension of the class of exact solutions of the equations of gas dynamics with shock waves is an interesting mathematical problem having various hydrodynamic applications.

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